

Hydrogen in Electrodynamics II. Mathematical Means

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After a discussion of the one-component Schrödinger (1926) and the four-component Dirac (1928) representation of hydrogen it is shown that the six-component electrodynamic picture turns out to be considerably simpler and clearer. The computational effort is reduced to a fraction.

For the hydrogen-solution of electrodynamics, as we shall see later, we need the simplest form in polar coordinates for the expressions

$$(\partial_1 + i \partial_2) \Psi = (\partial_1 + i \partial_2) C R(r) P_l^m(\cos \theta) e^{im\phi} \quad (1)$$

and $(C: \text{constant})$

$$\partial_3 \Psi = \partial_3 C R(r) P_l^m(\cos \theta) e^{im\phi}, \quad (2)$$

In order to obtain these two simplicia we consider the well-known operators

$$(\partial_1 \pm i \partial_2) = e^{\pm i\phi} \left(\sin \theta \partial_r + \frac{\cos \theta}{r} \partial_\theta \pm \frac{i}{r \sin \theta} \partial_\phi \right) \quad (3)$$

and

$$\partial_3 = \cos \theta \partial_r - \frac{\sin \theta}{r} \partial_\theta \quad (4)$$

as well as the associated spherical harmonics

$$P_l^m(\cos \theta) = \sin^m \theta \frac{d^m P_l(\cos \theta)}{d \cos^m \theta}, \quad 0 \leq |m| \leq l \quad (5)$$

In addition we need the well-known recurrence relations for the spherical harmonics and the simple representation of the trigonometric functions by spherical harmonics [1]:

$$(2l+1) \sin \theta P_l^m = (l+m-1)(l+m) P_{l-1}^{m-1} - (l-m+1)(l-m+2) P_{l+1}^{m-1}, \quad (6)$$

$$(2l+1) \left(\frac{m}{\sin \theta} P_l^m - \sin \theta \cos \theta d_{\cos \theta} P_l^m \right) = (l+1)(l+m)(l+m-1) P_{l-1}^{m-1} + l(l-m+1)(l-m+2) P_{l+1}^{m-1} \quad (7)$$

$$(2l+1) \sin^2 \theta d_{\cos \theta} P_l^m = (l+1)(l+m) P_{l-1}^m - l(l-m+1) P_{l+1}^m, \quad (8)$$

$$(2l+1) \cos \theta P_l^m = (l+m) P_{l-1}^m + (l-m+1) P_{l+1}^m, \quad (9)$$

$$\sin \theta = \frac{P_{l+1}^{m+1} - P_{l-1}^{m+1}}{(2l+1) P_l^m}, \quad (10)$$

$$\cos \theta = \frac{(l-m+1) P_{l+1}^m + (l+m) P_{l-1}^m}{(2l+1) P_l^m}. \quad (11)$$

If we replace m by $(-m)$ in (6) and (7) and employ the well-known relation [2]

$$P_l^{-m} = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m, \quad (12)$$

we then obtain the even simpler recurrence relations

$$(2l+1) \sin \theta P_l^m = -P_{l-1}^{m+1} + P_{l+1}^{m+1}, \quad (13)$$

$$(2l+1) \left(\frac{m}{\sin \theta} P_l^m + \sin \theta \cos \theta d_{\cos \theta} P_l^m \right) = (l+1) P_{l-1}^{m+1} + l P_{l+1}^{m+1}. \quad (14)$$

In order to finally obtain the desired form in polar coordinates for (1) we insert (3) and use (13) and (14) in the resulting expression. Then we get

$$(\partial_1 + i \partial_2) C R P_l^m e^{im\phi} = \frac{C e^{i(m+1)\phi}}{2l+1} \left(- \left(d_r + \frac{l+1}{r} \right) R P_{l-1}^{m+1} + \left(d_r - \frac{l}{r} \right) R P_{l+1}^{m+1} \right). \quad (15)$$

Taking into account also negative values of ϕ , this becomes

$$(\partial_1 \pm i \partial_2) C R P_l^m e^{\pm im\phi} = \frac{C e^{\pm i(m+1)\phi}}{2l+1} \left(- \left(d_r + \frac{l+1}{r} \right) R P_{l-1}^{m+1} + \left(d_r - \frac{l}{r} \right) R P_{l+1}^{m+1} \right). \quad (16)$$

For more clarity we introduce the operator

$$(d_r + a/r) R = R_{,a} \quad (a: \text{arbitrary}) \quad (17)$$

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and thus obtain for (1) the desired form

$$(\partial_1 \pm i \partial_2) CRP_l^m e^{\pm im\phi} = \frac{C e^{\pm i(m+1)\phi}}{2l+1} (-R_{,l+1} P_{l-1}^{m+1} + R_{,-l} P_{l+1}^{m+1}). \quad (18)$$

In order to obtain (2) also in an analogous form, we first insert from (4) and then from (8), (9) and (17). Then we get

$$\partial_3 CRP_l^m e^{im\phi} = \frac{C e^{im\phi}}{2l+1} [R_{,l+1}(l+m) P_{l-1}^m + R_{,-l}(l-m+1) P_{l+1}^m]. \quad (19)$$

Taking into account negative values of ϕ , we then obtain the desired relation also in this case

$$\partial_3 CRP_l^m e^{\pm im\phi} \quad (20)$$

$$= \frac{C e^{\pm im\phi}}{2l+1} [R_{,l+1}(l+m) P_{l-1}^m + R_{,-l}(l-m+1) P_{l+1}^m].$$

Moreover we note that the spherical harmonics have been generalized some time ago such that the most general form turns out to an entire function of m and l [3]. In such a generality the functions are therefore unique and continuous and can be differentiated after both indices. Halfinteger forms as, e.g.,

$$P_l^{m+1/2} = P_l^{m+1/2}(\cos \theta) \quad (21)$$

thus are uniquely defined.

[1] Proved in closed form in: Handbuch der Radiologie, Bd. VI, p.102, Leipzig 1933, – or in modern form in: H. A. Bauer, Grundlagen der Atomphysik, Springer, Wien 1951.

[2] K. Jänich, Analysis, p. 353, Springer, Berlin 1983.

[3] F. W. Schäfke, Einführung in die Theorie der speziellen Funktionen der Mathematischen Physik, Springer, Berlin 1963.