Hydrogen in Electrodynamics II. Mathematical Means

Hans Sallhofer

Forschungs- und Versuchsantalt
der AUSTRIA METALL AG, Braunau, Austria

Z. Naturforsch. **44a**, 167–168 (1989); received January 20, 1989

After a discussion of the one-component Schrödinger (1926) and the four-component Dirac (1928) represention of hydrogen it is shown that the six-component electrodynamic picture turns out to be considerably simpler and clearer. The computational effort is reduced to a fraction.

For the hydrogen-solution of electrodynamics, as we shall see later, we need the simplest form in polar coordinates for the expressions

$$(\partial_1 + i\,\partial_2)\,\Psi = (\partial_1 + i\,\partial_2)\,C\,R(r)\,P_l^m(\cos\theta)\,e^{im\phi} \tag{1}$$

and (C: constant)

$$\partial_3\,\Psi = \partial_3\,C\,R(r)\,P_l^m(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\phi}, \tag{2}$$

In order to obtain these two simplicia we consider the well-known operators

$$(\partial_1 \pm i\,\partial_2) = e^{\pm i\phi}\left(\sin\theta\,\partial_r + \frac{\cos\theta}{r}\,\partial_\theta \pm \frac{i}{r\sin\theta}\,\partial_\phi\right) \tag{3}$$

and

$$\partial_3 = \cos\theta\,\partial_r - \frac{\sin\theta}{r}\,\partial_\theta \tag{4}$$

as well as the associated spherical harmonics

$$P_l^m(\cos\theta) = \sin^m\theta\,\frac{\mathrm{d}^m\,P_l(\cos\theta)}{\mathrm{d}\cos^m\theta},\quad 0 \leqq |m| \leqq l \tag{5}$$

In addition we need the well-known recurrence relations for the spherical harmonics and the simple representation of the trigonometric functions by spherical harmonics [1]:

$$(2l+1)\sin\theta\,P_l^m \tag{6}$$

$$= (l+m-1)(l+m)\,P_{l-1}^{m-1} - (l-m+1)(l-m+2)\,P_{l+1}^{m-1},$$

$$(2l+1)\left(\frac{m}{\sin\theta}\,P_l^m - \sin\theta\cos\theta\,d_{\cos\theta}\,P_l^m\right)$$

$$= (l+1)(l+m)(l+m-1)\,P_{l-1}^{m-1} + l(l-m+1)(l-m+2)\,P_{l+1}^{m-1} \tag{7}$$

$$(2l+1)\sin^2\theta\,d_{\cos\theta}\,P_l^m$$

$$= (l+1)(l+m)\,P_{l-1}^m - l(l-m+1)\,P_{l+1}^m, \tag{8}$$

$$(2l+1)\cos\theta\,P_l^m = (l+m)\,P_{l-1}^m + (l-m+1)\,P_{l+1}^m, \tag{9}$$

$$\sin\theta = \frac{P_{l+1}^{m+1} - P_{l-1}^{m+1}}{(2l+1)\,P_l^m}, \tag{10}$$

$$\cos\theta = \frac{(l-m+1)\,P_{l+1}^m + (l+m)\,P_{l-1}^m}{(2l+1)\,P_l^m}. \tag{11}$$

If we replace m by $(-m)$ in (6) and (7) and employ the well-known relation [2]

$$P_l^{-m} = (-1)^m\,\frac{(l-m)!}{(l+m)!}\,P_l^m, \tag{12}$$

we then obtain the even simpler recurrence relations

$$(2l+1)\sin\theta\,P_l^m = -P_{l-1}^{m+1} + P_{l+1}^{m+1}, \tag{13}$$

$$(2l+1)\left(\frac{m}{\sin\theta}\,P_l^m + \sin\theta\cos\theta\,d_{\cos\theta}\,P_l^m\right)$$

$$= (l+1)\,P_{l-1}^{m+1} + l\,P_{l+1}^{m+1}. \tag{14}$$

In order to finally obtain the desired form in polar coordinates for (1) we insert (3) and use (13) and (14) in the resulting expression. Then we get

$$(\partial_1 + i\,\partial_2)\,CRP_l^m\,e^{im\phi}$$

$$= \frac{C\,e^{\mathrm{i}(m+1)\phi}}{2l+1}\left(-\left(d_r + \frac{l+1}{r}\right)RP_{l-1}^{m+1} + \left(d_r - \frac{l}{r}\right)RP_{l+1}^{m+1}\right). \tag{15}$$

Taking into account also negative values of ϕ, this becomes

$$(\partial_1 \pm i\,\partial_2)\,CRP_l^m\,e^{\pm im\phi}$$

$$= \frac{C\,e^{\pm i(m+1)\phi}}{2l+1}\left(-\left(d_r + \frac{l+1}{r}\right)RP_{l-1}^{m+1} + \left(d_r - \frac{l}{r}\right)RP_{l+1}^{m+1}\right). \tag{16}$$

For more clarity we introduce the operator

$$(d_r + a/r)\,R = R_{,a} \quad (a\text{: arbitrary}) \tag{17}$$

Reprint requests to Prof. Dr. H. Sallhofer, Bahnhofstrasse 36, A-5280 Braunau, Austria.

0932-0784 / 89 / 0200-0167 $ 01.30/0. – Please order a reprint rather than making your own copy.

and thus obtain for (1) the desired form

$(\partial_1 \pm i\,\partial_2)\, CRP_l^m\, e^{\pm i m\phi}$

$$= \frac{C\, e^{\pm i(m+1)\phi}}{2l+1}\,(-R_{,l+1}\, P_{l-1}^{m+1} + R_{,-l}\, P_{l+1}^{m+1})\,. \quad (18)$$

In order to obtain (2) also in an analogous form, we first insert from (4) and then from (8), (9) and (17). Then we get

$$\partial_3\, CRP_l^m\, e^{i m\phi} \quad (19)$$

$$= \frac{C\, e^{i m\phi}}{2l+1}\,[R_{,l+1}(l+m)\, P_{l-1}^m + R_{,-l}(l-m+1)\, P_{l+1}^m]\,.$$

Taking into account negative values of ϕ, we then obtain the desired relation also in this case

$$\partial_3\, CRP_l^m\, e^{\pm i m\phi} \quad (20)$$

$$= \frac{C\, e^{\pm i m\phi}}{2l+1}\,[R_{,l+1}(l+m)\, P_{l-1}^m + R_{,-1}(l-m+1)\, P_{l+1}^m]\,.$$

Moreover we note that the spherical harmonics have been generalized some time ago such that the most general form turns out to an entire function of m and l [3]. In such a generality the functions are therefore unique and continuous and can be differentiated after both indices. Halfinteger forms as, e.g.,

$$P_l^{m+1/2} = P_l^{m+1/2}(\cos\theta) \quad (21)$$

thus are uniquely defined.

[1] Proved in closed form in: Handbuch der Radiologie, Bd. VI, p. 102, Leipzig 1933, – or in modern form in: H. A. Bauer, Grundlagen der Atomphysik, Springer, Wien 1951.
[2] K. Jänich, Analysis, p. 353, Springer, Berlin 1983.
[3] F. W. Schäfke, Einführung in die Theorie der speziellen Funktionen der Mathematischen Physik, Springer, Berlin 1963.

Verantwortlich für den Inhalt: A. KLEMM
Satz und Druck: Konrad Triltsch, Würzburg